

FINITE DIGRAPHS AND KMS STATES

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1. INTRODUCTION

In a recent paper by an Huef, Laca, Raeburn and Sims, [aHLRS], the authors describe an algorithm by which it is possible to determine all the KMS states of the gauge action on the C^* -algebra of a finite graph. Their results cover also the gauge action on the Toeplitz extension of the algebra and extend the result of Enomoto, Fujii and Watatani, [EFW], which deals with strongly connected graphs. Almost simultaneously with this work, Carlsen and Larsen obtained an abstract description of the KMS states for some of the generalized gauge actions on the C^* -algebra of a finite graph as well as its Toeplitz extension. Their work build on and extend methods and results obtained by Exel and Laca in [EL] and bring our knowledge about the KMS states of the actions they consider to the point where the work on the gauge action begins in [aHLRS]. It is the purpose of the present paper to take the steps from the abstract to the concrete which were taken by an Huef, Laca, Raeburn and Sims, but now for all the generalized gauge actions.

The point of departure for our work are results of the second author from [Th3] from which it follows that the relevant results of Carlsen and Larsen from [CL] remain valid for all generalized gauge actions, provided attention is restricted to the KMS states that are gauge invariant; a condition which is automatically satisfied for the actions considered by Carlsen and Larsen. What we do first is to develop the approach from [aHLRS] to make it applicable to generalized gauge actions. In this way we obtain a description of the gauge invariant KMS states for all generalized gauge actions. The main input for this is a generalization of the Perron-Frobenius theory for positive matrices which was obtained by Victory, [Vi]. See also [Ta]. The theory handles arbitrary finite non-negative matrices and can also be used to simplify some of the steps in [aHLRS]. We give here a new proof of the relevant results from [Vi] and [Ta] by using ideas from [aHLRS].

In order to identify the KMS states that are not gauge invariant we use results by Neshveyev, [N], in a form presented in [Th1]. By combining the result with our study of the gauge invariant KMS states we obtain in Theorem 5.2 our main result which describes the β -KMS states for all $\beta \in \mathbb{R} \setminus \{0\}$ and for an arbitrary generalized gauge action on the C^* -algebra of a finite graph. As with the gauge action, [aHLRS], it is a sub-collection of the components and the sinks in the graph that parametrize the extremal KMS states, although in general some of the components, corresponding to a loop without exits, may contribute a family of extremal KMS states parametrized by a circle. Which components and sinks play a role depends very much on the action, as we illustrate by examples.

It is intrinsic to our approach that the case $\beta = 0$, where the KMS states are the trace states, must be handled separately as we do in Section 5.1. For completeness

we describe also in a final section the ground states for the same actions. While there are no ground states for the gauge action unless the graph has sinks, this is no longer the case for generalized gauge actions and even for strongly connected graphs their structure can be very rich.

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2. PREPARATIONS

Let G be a finite directed graph with vertex set V and edge set E . The maps $r, s : E \rightarrow V$ associate to an edge $e \in E$ its source vertex $s(e) \in V$ and range vertex $r(e) \in V$. Thus the set of edges emitted from a vertex v is the set $s^{-1}(v)$ while $r^{-1}(v)$ is the set of edges terminating at v . A *sink* in G is a vertex v that does not emit an edge, i.e. $s^{-1}(v) = \emptyset$.

Formulated in terms of generators and relations the C^* -algebra $C^*(G)$ of G is the universal C^* -algebra generated by a set $S_e, e \in E$, of partial isometries and a set $P_v, v \in V$, of mutually orthogonal projections such that

$$\begin{aligned} 1) \quad & S_e^* S_e = P_{r(e)} \quad \forall e \in E, \text{ and} \\ 2) \quad & P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* \text{ for every vertex } v \in V \text{ which is not a sink.} \end{aligned} \quad (2.1)$$

A finite path μ in G is an element $\mu = e_1 e_2 \cdots e_n \in E^n$ for some $n \in \mathbb{N}$ such that $r(e_i) = s(e_{i+1}), i = 1, 2, \dots, n-1$. For such a path we set

$$S_\mu = S_{e_1} S_{e_2} \cdots S_{e_{n-1}} S_{e_n}.$$

The number $|\mu| = n$ is the length of the path. We consider a vertex v as a path ν of length 0, and set $S_\nu = P_v$ in this case. Let $P_f(G)$ denote the set of finite paths in G . Then

$$\mathcal{A} = \text{Span} \{ S_\mu S_\nu^* : \mu, \nu \in P_f(G) \} \quad (2.2)$$

is a dense $*$ -subalgebra of $C^*(G)$.

Let $F : E \rightarrow \mathbb{R}$ be a function. The universal property of $C^*(G)$ guarantees the existence of a one-parameter group $\alpha_t^F, t \in \mathbb{R}$, on $C^*(G)$ such that

$$\alpha_t^F(P_v) = P_v \quad \forall v \in V, \quad \text{and} \quad \alpha_t^F(S_e) = e^{iF(e)t} S_e \quad \forall e \in E.$$

For $\beta \in \mathbb{R}$ a β -KMS state for α^F is a state φ on $C^*(G)$ such that

$$\varphi(ab) = \varphi(b \alpha_{i\beta}^F(a))$$

for all $a, b \in \mathcal{A}$, cf. Definition 5.3.1 in [BR]. When F is constant 1 the automorphism group $\{\alpha_t^1\}$ is the so-called *gauge action* and we study first the gauge invariant KMS states for α^F , i.e the KMS states φ for α^F with the additional property that $\varphi \circ \alpha_t^1 = \varphi$ for all $t \in \mathbb{R}$. For this purpose we use the following description of the gauge invariant KMS states. It was obtained by Carlsen and Larsen in [CL] when F is strictly positive (in which case all KMS states for α^F are gauge-invariant). The general case follows from Theorem 2.8 in [Th3].

Let B be a non-negative matrix over V with the property that $B_{vw} > 0$ iff there is an edge in G from v to w . A vector $\psi \in [0, \infty)^V$ is *almost harmonic* for B (or *almost B-harmonic*) when

$$\sum_{w \in V} B_{vw} \psi_w = \psi_v \quad (2.3)$$

for every vertex $v \in V$ which is not a sink, and *normalized* when $\sum_{v \in V} \psi_v = 1$. When the identity (2.3) holds for all $v \in V$ we say that ψ is *harmonic* for B (or *B-harmonic*). Thus an almost B -harmonic vector ψ is B -harmonic if and only if $\psi_s = 0$ for every sink $s \in V$. For $\beta \in \mathbb{R}$, consider the matrix $A(\beta) = (A(\beta)_{vw})$ over V defined such that

$$A(\beta)_{vw} = \sum_{e \in vEw} e^{-\beta F(e)},$$

where $vEw = s^{-1}(v) \cap r^{-1}(w)$. For a finite path $\mu = e_1 e_2 \cdots e_n$ in G , set

$$F(\mu) = F(e_1) + F(e_2) + \cdots + F(e_n).$$

Lemma 2.1. ([CL], [Th3]) *For every normalized $A(\beta)$ -almost harmonic vector ψ there is a unique gauge invariant β -KMS state ω_ψ for α^F such that*

$$\omega_\psi(S_\mu S_\nu^*) = \delta_{\mu,\nu} e^{-\beta F(\mu)} \psi_{r(\mu)} \quad (2.4)$$

for every pair μ, ν of finite paths in G . Furthermore, every gauge invariant β -KMS state for α^F arises from a normalized $A(\beta)$ -almost harmonic vector in this way.

By Lemma 2.1 the study of the gauge invariant KMS states becomes a study of normalized almost harmonic vectors for the family $A(\beta), \beta \in \mathbb{R}$, of non-negative matrices over V .

3. ALMOST HARMONIC VECTORS FOR A NON-NEGATIVE MATRIX

Let B be a non-negative matrix over V with the property that $B_{vw} > 0$ iff there is an edge in G from v to w . We seek to obtain a description of the B -almost harmonic vectors.

We shall need the following well-known lemma, cf. e.g. 6.43 in [W].

Lemma 3.1. (*Riesz decomposition.*) *Let $\psi = (\psi_v)_{v \in V} \in [0, \infty]^V$ be a non-negative vector such that*

$$\sum_{w \in V} B_{vw} \psi_w \leq \psi_v$$

for all $v \in V$. It follows that there are unique non-negative vectors $h, k \in [0, \infty]^V$ such that h is B -harmonic and

$$\psi_v = h_v + \sum_{w \in V} \sum_{n=0}^{\infty} B_{vw}^n k_w \quad (3.1)$$

for all $v \in V$. The vector k is given by

$$k_v = \psi_v - \sum_{w \in V} B_{vw} \psi_w,$$

while

$$h_v = \lim_{n \rightarrow \infty} \sum_{w \in V} B_{vw}^n \psi_w.$$

We say that a sink $s \in V$ is *B-summable* when

$$\sum_{n=0}^{\infty} B_{vs}^n < \infty$$

for all $v \in V$. For such a sink we define a vector $\phi^s \in [0, \infty)^V$ by

$$\phi_v^s = \frac{\sum_{n=0}^{\infty} B_{vs}^n}{\sum_{w \in V} \sum_{n=0}^{\infty} B_{ws}^n}.$$

Lemma 3.2. ϕ^s in an extremal normalized B -almost harmonic vector.

Proof. The only assertion which may not be straightforward to verify is that ϕ^s is extremal in the set of normalized B -almost harmonic vectors. To show this, consider a B -almost harmonic vector φ with the property that $\varphi \leq \phi^s$. Since

$$B_{vw}^m \varphi_w \leq B_{vw}^m \phi_w^s \leq \frac{\sum_{n=m}^{\infty} B_{vs}^n}{\sum_{w \in V} \sum_{n=0}^{\infty} B_{ws}^n} \rightarrow 0 \quad (3.2)$$

as $m \rightarrow \infty$, it follows that the harmonic part from the Riesz decomposition of φ is zero. Thus

$$\varphi_v = \sum_{w \in V} \sum_{n=0}^{\infty} B_{vw}^n k_w$$

where $k_v = \varphi_v - \sum_{w \in V} B_{vw} \varphi_w$. Note that $k_v = 0$ when v is not a sink since φ is B -almost harmonic, and that $k_{s'} = \varphi_{s'}$ for every sink s' . Note also that $\phi_{s'}^s = 0$ for every sink s' in G other than s . Since $\varphi \leq \phi^s$ it follows that the same is true for φ . Hence

$$\varphi_v = \sum_{n=0}^{\infty} B_{vs}^n \varphi_s = t \phi_v^s,$$

where

$$t = \varphi_s \sum_{w \in V} \sum_{n=0}^{\infty} B_{ws}^n.$$

□

By combining Lemma 3.1 and Lemma 3.2 we obtain the following

Proposition 3.3. *Let ψ be a normalized B -almost harmonic vector. There are a unique (possibly empty) set \mathcal{S} of summable sinks in G , unique positive numbers $t_s \in]0, 1]$, $s \in \mathcal{S}$, and a unique B -harmonic vector h such that*

$$\psi = h + \sum_{s \in \mathcal{S}} t_s \phi^s.$$

We turn to a study of the B -harmonic vectors. For any pair of subsets $E, D \subseteq V$ we let $B^{E,D}$ denote the $E \times D$ -matrix obtained by restricting B to $E \times D$, and we set $B^E = B^{E,E}$ for any subset $E \subseteq V$.

Write $v \rightsquigarrow w$ between two vertexes v, w when there is a finite path $\mu = e_1 \cdots e_n$ in G such that $s(e_1) = v$ and $r(e_n) = w$, and $v \sim w$ when $v \rightsquigarrow w$ and $w \rightsquigarrow v$. Then \sim is an equivalence relation since we consider a vertex v as a finite path (of length 0) from v to v . A *component* C in G is an equivalence class in V/\sim such that $B^C \neq 0$. For any collection F of vertexes in G we define the *closure* of F to be the set of vertexes that 'talk' to an element of F , i.e. $v \in \overline{F}$ if and only if there is a vertex $w \in F$ such that $v \rightsquigarrow w$. In contrast the *hereditary closure* of a set F consists of the vertexes $w \in V$ such that $v \rightsquigarrow w$ for some $v \in F$. The hereditary closure will be denoted by \widehat{F} .

In the following we denote the spectral radius of a finite matrix A by $\rho(A)$. A component C in G is *B -harmonic* when

- a) $\rho(B^C) = 1$ and
- b) $\rho(B^{\overline{C} \setminus C}) < 1$ if $\overline{C} \setminus C \neq \emptyset$.

This definition, as well as the proof of the following lemma, is inspired by Theorem 4.3 in [aHLRS].

Lemma 3.4. *Let C be a B -harmonic component in G . There is a unique normalized B -harmonic vector ϕ^C such that $B^C \phi^C|_C = \phi^C|_C$ and $\phi_v^C \neq 0 \Leftrightarrow v \in \overline{C}$.*

Proof. Existence: Since $\rho(B^C) = 1$ it follows from Perron-Frobenius theory that there is a strictly positive vector $x^C \in [0, \infty)^C$ such that $B^C x^C = x^C$. Since $\rho(B^{\overline{C} \setminus C}) < 1$, the matrix $1^{\overline{C} \setminus C} - B^{\overline{C} \setminus C}$ is invertible and we set

$$\phi^C = \left(1^{\overline{C} \setminus C} - B^{\overline{C} \setminus C}\right)^{-1} B^{\overline{C} \setminus C, C} x^C + x^C,$$

which is a strictly positive vector in $[0, \infty)^{\overline{C}}$. For any pair of vertexes $v, w \in \overline{C} \setminus C$,

$$\limsup_n (B_{v,w}^n)^{\frac{1}{n}} \leq \rho(B^{\overline{C} \setminus C}) < 1,$$

and hence

$$\left(1^{\overline{C} \setminus C} - B^{\overline{C} \setminus C}\right)^{-1} = \sum_{n=0}^{\infty} \left(B^{\overline{C} \setminus C}\right)^n.$$

Using this and that no vertex in C talks to a vertex in $\overline{C} \setminus C$, we find that

$$\begin{aligned} B^{\overline{C}} \phi^C &= B^{\overline{C} \setminus C} \left(1^{\overline{C} \setminus C} - B^{\overline{C} \setminus C}\right)^{-1} B^{\overline{C} \setminus C, C} x^C + B^{\overline{C} \setminus C, C} x^C + B^C x^C \\ &= B^{\overline{C} \setminus C} \sum_{n=0}^{\infty} \left(B^{\overline{C} \setminus C}\right)^n B^{\overline{C} \setminus C, C} x^C + B^{\overline{C} \setminus C, C} x^C + x^C \\ &= \sum_{n=1}^{\infty} \left(B^{\overline{C} \setminus C}\right)^n B^{\overline{C} \setminus C, C} x^C + B^{\overline{C} \setminus C, C} x^C + x^C \\ &= \sum_{n=0}^{\infty} \left(B^{\overline{C} \setminus C}\right)^n B^{\overline{C} \setminus C, C} x^C + x^C \\ &= \phi^C. \end{aligned} \tag{3.3}$$

Set $\phi_v^C = 0$ when $v \notin \overline{C}$ and normalize the resulting vector in $[0, \infty)^V$. It follows from (3.3) that ϕ^C is B -harmonic. Since $\phi^C|_C$ is multiple of x^C by construction, it follows that $B^C \phi^C|_C = \phi^C|_C$.

Uniqueness: If ψ is a normalized B -harmonic vector such that $B^C \psi|_C = \psi|_C$ and $\psi_v \neq 0 \Leftrightarrow v \in \overline{C}$, it follows from Perron-Frobenius theory that there is a $\lambda > 0$ such that $\psi_v = \lambda \phi_v^C \forall v \in C$. Then $\psi - \lambda \phi^C$ is vector supported in $\overline{C} \setminus C$ such that $B^{\overline{C} \setminus C}(\psi - \lambda \phi^C) = \psi - \lambda \phi^C$. Since $\rho(B^{\overline{C} \setminus C}) < 1$, it follows first that $\psi = \lambda \phi^C$ and then that $\psi = \phi^C$ because both vectors are normalized. \square

The following theorem is equivalent to the Frobenius-Victory theorem stated as Theorem 2.7 in [Ta].

Theorem 3.5. *Let $\psi \in [0, 1]^V$ be a normalized B -harmonic vector. There is a unique collection \mathcal{C} of B -harmonic components in G and positive numbers $t_C \in]0, 1]$, $C \in \mathcal{C}$, such that*

$$\psi = \sum_{C \in \mathcal{C}} t_C \phi^C. \quad (3.4)$$

Proof. Set $\odot = \{v \in V : \psi_v > 0\}$. Let $v \in \odot$. Since $B_{vv}^n \psi_v \leq \psi_v$ for all n , it follows that

$$\limsup_n (B_{vv}^n)^{\frac{1}{n}} \leq 1.$$

Hence

$$\rho(B^\odot) = \sup_{v \in \odot} \limsup_n (B_{vv}^n)^{\frac{1}{n}} \leq 1.$$

On the other hand, the fact that $B^\odot \psi|_\odot = \psi|_\odot$ implies that $\rho(B^\odot) \geq 1$, and we conclude that

$$\rho(B^\odot) = 1. \quad (3.5)$$

Since

$$\rho(B^\odot) = \sup_C \rho(B^C),$$

where we take the supremum over the components of G contained in \odot , the collection \mathcal{C}' of components C from G such that $C \subseteq \odot$ and $\rho(B^C) = 1$ is not empty. Order the elements of \mathcal{C}' such that $C \leq C'$ when the elements in C talk to the elements of C' . Let \mathcal{C} be the minimal elements of \mathcal{C}' with respect to this order. Let $D \in \mathcal{C}$. We claim that D is a B -harmonic component, i.e. we assert that

$$\rho(B^{\overline{D} \setminus D}) < 1.$$

Since $\overline{D} \subseteq \odot$ it follows from (3.5) that $\rho(B^{\overline{D} \setminus D}) \leq 1$. If $\rho(B^{\overline{D} \setminus D}) = 1$, there must be one of G 's components, say D' , contained in $\overline{D} \setminus D$ such that $\rho(B^{D'}) = 1$. But then $D' \in \mathcal{C}'$, $D' \neq D$ and $D' \leq D$, contradicting the minimality of D . Hence D is B -harmonic as claimed, and we conclude that \mathcal{C} consists of B -harmonic components.

Let $D \in \mathcal{C}$. Then $B^D \psi|_D \leq \psi|_D$ so it follows from Perron-Frobenius theory that there is $t_D \geq 0$ such that $\psi|_D = t_D \psi^D|_D$. Since $\psi|_D$ and $\psi^D|_D$ are strictly positive, t_D is positive too. Set

$$\eta = \psi - \sum_{D \in \mathcal{C}} t_D \psi^D.$$

We claim that $\eta = 0$. To show this, set $K = \bigcup_{D \in \mathcal{C}} D$, and note that $\eta|_K = 0$. Let H be the hereditary closure of K , i.e. $H = \widehat{K}$. Consider a $D \in \mathcal{C}$. When $v \in (H \setminus K) \cap \overline{D}$, there is a path from (some element of) $D' \subseteq K$ to v and a path from v to (some element of) D . Note that $D' \neq D$ since otherwise v would have to be an element of $D \subseteq K$. But $D' \neq D$ is impossible since D is minimal for the order on \mathcal{C}' . Hence $(H \setminus K) \cap \overline{D} = \emptyset$, showing that $\psi^D|_{H \setminus K} = 0$. It follows that $\eta|_{H \setminus K} = \psi|_{H \setminus K}$, and hence that $\eta|_H \geq 0$. Let $w \in H$. There is an $l \in \mathbb{N}$ and $v \in K$ such that $B_{vw}^l \neq 0$. Since $B^l \eta = \eta$ we find that $0 = \eta_v = \sum_{u \in V} B_{vu}^l \eta_u \geq B_{vw}^l \eta_w \geq 0$, implying that $\eta_w = 0$. Hence $\eta|_H = 0$. Now note that

$$\rho(B^{\odot \setminus H}) < 1 \quad (3.6)$$

since all components D in \mathbb{C} with $\rho(B^D) = 1$ are contained in H . Since

$$(B^{\mathbb{C} \setminus H} \eta)_v = \sum_{w \in \mathbb{C} \setminus H} B_{vw} \eta_w = \sum_{w \in V} B_{vw} \eta_w = \eta_v$$

for all $v \in \mathbb{C} \setminus H$, it follows from (3.6) that $\eta|_{\mathbb{C} \setminus H} = 0$. Thus $\eta = 0$ as claimed and (3.4) follows.

To prove the uniqueness part of the statement let \mathcal{D} be a collection of B -harmonic components in G and $s_C, C \in \mathcal{D}$, positive numbers such that

$$\psi = \sum_{C \in \mathcal{D}} s_C \phi^C.$$

Then $\mathbb{C} = \bigcup_{C \in \mathcal{C}} \overline{C} = \bigcup_{C \in \mathcal{D}} \overline{C}$, so when $C \in \mathcal{D}$ there is a $C' \in \mathcal{C}$ such that $C \cap \overline{C'} \neq \emptyset$. It follows that $C \subseteq \overline{C'}$ and that either $C' = C$ or $C \subseteq \overline{C'} \setminus C'$. However, $\rho(B^C) = 1$ while $\rho(B^{\overline{C'} \setminus C'}) < 1$, and it follows therefore that $C = C'$. In this way we conclude that $\mathcal{D} = \mathcal{C}$. Since the preceding argument shows that $C \cap \overline{C'} = \emptyset$ when C and C' are distinct elements from \mathcal{C} , we find that

$$s_C \phi^C|_C = \psi|_C = t_C \phi^C|_C,$$

and hence that $s_C = t_C$ for all $C \in \mathcal{C}$. □

Corollary 3.6. *The normalized B -harmonic vectors constitute a finite dimensional simplex whose set of extreme points is*

$$\{\phi^C : C \text{ a } B\text{-harmonic component in } G\}.$$

Combining Theorem 3.5 with Proposition 3.3 we obtain the following

Corollary 3.7. *The set of normalized B -almost harmonic vectors constitute a finite dimensional simplex whose set of extreme points is*

$$\{\phi^C : C \text{ a } B\text{-harmonic component in } G\} \cup \{\phi^s : s \text{ a } B\text{-summable sink in } G\}.$$

4. GAUGE INVARIANT KMS STATES

It follows from Lemma 2.1 and Corollary 3.7 that the gauge invariant β -KMS states for α^F are determined by the $A(\beta)$ -harmonic components and the $A(\beta)$ -summable sinks. In this section we complete the description of the gauge invariant KMS states for $\beta \neq 0$ by finding the $A(\beta)$ -harmonic components and the $A(\beta)$ -summable sinks for each $\beta \in \mathbb{R} \setminus \{0\}$.¹

4.1. $A(\beta)$ -harmonic components. A *loop* in G is a finite path $\mu = e_1 e_2 \cdots e_n$ (of positive length, i.e. $n \geq 1$) such that $s(e_1) = r(e_n)$. If a component C only contains a single loop, we call it *circular*.

Lemma 4.1. *Let $C \subseteq V$ be a component. The function*

$$\mathbb{R} \ni \beta \mapsto \rho(A(\beta)^C)$$

is log-convex and continuous.

¹We could have handled the case $\beta = 0$ here also, but it does simplify things a little when $\beta \neq 0$, and we will have to consider the $\beta = 0$ case separately for other reasons anyway.

Proof. Since C is a component there is a loop in C , of length p , say. Let v be a vertex on this loop. It follows that $\log \rho(A(\beta)^C) \geq \frac{1}{p} \log(A(\beta)^C)_{vv}^p$, showing that the logarithm of the function we consider takes finite values for all β . Its continuity follows therefore from its log-convexity which is established as follows. Let $v \in C$ and $\beta, \beta' \in \mathbb{R}, t \in [0, 1]$. For each $n \in \mathbb{N}$ let $vE^n v$ denote the set of paths of length n from v back to itself. Then

$$(A(t\beta + (1-t)\beta')^C)_{vv}^n = \sum_{\mu \in vE^n v} e^{-(t\beta + (1-t)\beta')F(\mu)} = \sum_{\mu \in vE^n v} (e^{-\beta F(\mu)})^t (e^{-\beta' F(\mu)})^{1-t}.$$

Then Hölders inequality shows that

$$(A(t\beta + (1-t)\beta')^C)_{vv}^n \leq ((A(\beta)^C)_{vv}^n)^t ((A(\beta')^C)_{vv}^n)^{1-t}.$$

It follows that

$$\rho(A(t\beta + (1-t)\beta')^C) = \limsup_n ((A(t\beta + (1-t)\beta')^C)_{vv}^n)^{\frac{1}{n}}$$

is dominated by the product

$$\rho(A(\beta)^C)^t \rho(A(\beta')^C)^{1-t},$$

which is what we needed to prove. \square

Lemma 4.2. *Let C be a component in G which is not circular.*

- i) *If $F(\mu) > 0$ for all loops μ in C , there is a unique $\beta_0 \in \mathbb{R}$ such that $\rho(A(\beta_0)^C) = 1$. This β_0 is positive and $\rho(A(\beta)^C) < 1$ if and only if $\beta > \beta_0$.*
- ii) *If $F(\mu) < 0$ for all loops μ in C , there is a unique $\beta_0 \in \mathbb{R}$ such that $\rho(A(\beta_0)^C) = 1$. This β_0 is negative and $\rho(A(\beta)^C) < 1$ if and only if $\beta < \beta_0$.*
- iii) *In all other cases, i.e. if $F(\mu) = 0$ for some loop in C or there are loops μ_1, μ_2 in C such that $F(\mu_1) < 0 < F(\mu_2)$, it follows that $\rho(A(\beta)^C) > 1$ for all $\beta \in \mathbb{R}$.*

Proof. Some of the following arguments have appeared in [Th3]. i): We claim that $\beta \mapsto \rho(A(\beta)^C)$ is strictly decreasing. To see this, set

$$a = \min \{F(\mu) : \mu \text{ is a loop in } C \text{ of length } |\mu| \leq \#C\}.$$

Consider $\beta' < \beta$ and a loop μ in C of length n . Then $\mu = \mu_1 \mu_2 \cdots \mu_m$, where each μ_i is a loop in C of length $\leq \#C$, and

$$e^{-\beta' F(\mu)} e^{\beta F(\mu)} = \prod_j e^{(\beta - \beta') F(\mu_j)} \geq e^{m(\beta - \beta')a} \geq e^{\frac{n}{\#C}(\beta - \beta')a}.$$

Summing over all loops of length n starting and ending at the same vertex v in C , it follows first that

$$(A(\beta')^C)_{vv}^n \geq e^{\frac{n}{\#C}(\beta - \beta')a} (A(\beta)^C)_{vv}^n,$$

and then that

$$\rho(A(\beta')^C) = \limsup_n ((A(\beta')^C)_{vv}^n)^{\frac{1}{n}} \geq \rho(A(\beta)^C) e^{\frac{1}{\#C}(\beta - \beta')a} > \rho(A(\beta)^C).$$

This proves the claim. Note that $A(0)^C$ is the adjacency matrix of the subgraph H of G whose vertex set is C . This is a finite strongly connected graph and it is well-known, and easy to show, that $\rho(A(0)^C) > 1$ because H by assumption consists of more than a single loop. In view of Lemma 4.1 it suffices now to show that

$\lim_{\beta \rightarrow \infty} \rho(A(\beta)^C) = 0$. To this end note that any path in H of length $\geq \#C$ must visit at least one vertex twice. It follows that for any path $\mu \in P_f(H)$ of length n with $r(\mu) = s(\mu)$ there is a finite collection

$$\{\nu_1, \nu_2, \dots, \nu_N\} \subseteq \{\mu \in P_f(H) : 1 \leq |\mu| \leq \#C, s(\mu) = r(\mu)\}$$

such that $N \geq \frac{n}{\#C}$ and

$$F(\mu) = \sum_{j=1}^N F(\nu_j) \geq Na \geq \frac{na}{\#C}.$$

Let $\beta > 0$ and $v \in C$. Then

$$A(\beta)_{vv}^n = \sum_{\mu \in vE^{nv}} e^{-\beta F(\mu)} \leq A(0)_{vv}^n e^{-\frac{\beta an}{\#C}},$$

Hence

$$\rho(A(\beta)^C) = \limsup_n ((A(\beta)^C)_{vv}^n)^{\frac{1}{n}} \leq \rho(A(0)^C) e^{-\frac{\beta a}{\#C}}.$$

Since $a > 0$, it follows that $\lim_{\beta \rightarrow \infty} \rho(A(\beta)^C) = 0$.

The proof of ii) is analogous to that of i).

iii): Assume first that $F(\mu) = 0$ for some loop in C . Since we assume that C is not circular, there is a path ν such that $|\nu| = m|\mu|$ for some $m \in \mathbb{N}$, $s(\nu) = r(\nu) = s(\mu)$ and ν is not the composition of m copies of μ . It follows that, with $v = s(\mu)$,

$$(A(\beta)^C)_{vv}^{nm|\mu|} \geq \left((A(\beta)^C)_{vv}^{m|\mu|} \right)^n \geq (e^{-\beta m F(\mu)} + e^{-\beta F(\nu)})^n = (1 + e^{-\beta F(\nu)})^n$$

for all $n \in \mathbb{N}$, showing that

$$\rho(A(\beta)^C) \geq (1 + e^{-\beta F(\nu)})^{\frac{1}{m|\mu|}} > 1$$

for all $\beta \in \mathbb{R}$. Assume then that there are loops μ_1, μ_2 in C such that $F(\mu_1) < 0 < F(\mu_2)$. We may assume that μ_1 and μ_2 start at the same vertex v , if necessary after a modification of μ_1 or μ_2 . Then

$$(A(\beta)^C)_{vv}^{n|\mu_1||\mu_2|} \geq \max \{e^{-\beta n|\mu_2|F(\mu_1)}, e^{-\beta n|\mu_1|F(\mu_2)}\}$$

for all $n \in \mathbb{N}$, proving that

$$\rho(A(\beta)^C) \geq \max \left\{ e^{-\beta \frac{F(\mu_1)}{|\mu_1|}}, e^{-\beta \frac{F(\mu_2)}{|\mu_2|}} \right\} > 1$$

for all $\beta \neq 0$. This completes the proof because $\rho(A(0)^C) > 1$ since C is not circular. \square

Lemma 4.3. *Let C be a circular component consisting of the vertexes in the loop μ . Then*

$$\rho(A(\beta)^C) = e^{-\beta \frac{F(\mu)}{|\mu|}}$$

for all $\beta \in \mathbb{R}$.

Proof. Left to the reader. \square

Let C be a component. It follows from Lemma 4.2 and Lemma 4.3 that when $F(\mu) > 0$ for every loop μ in C , or $F(\mu) < 0$ for every loop in C , there is a unique number $\beta_C \in \mathbb{R}$ such that

$$\rho(A(\beta_C)^C) = 1.$$

Definition 4.4. A non-circular component C in G is a *KMS component of positive type* when

- i) $F(\mu) > 0$ for every loop μ in \overline{C} , and
- ii) $\beta_{C'} < \beta_C$ for every component C' in $\overline{C} \setminus C$, if any.

Similarly, a non-circular component C in G is a *KMS component of negative type* when

- i) $F(\mu) < 0$ for every loop μ in \overline{C} , and
- ii) $\beta_C < \beta_{C'}$ for every component C' in $\overline{C} \setminus C$, if any.

Lemma 4.5. i) Let $\beta > 0$. A non-circular component C is $A(\beta)$ -harmonic if and only if C is a KMS component of positive type and $\beta_C = \beta$.
 ii) Let $\beta < 0$. A non-circular component C is $A(\beta)$ -harmonic if and only if C is a KMS component of negative type and $\beta_C = \beta$.

Proof. The proofs of the two cases are identical and we consider here only case i): By definition, C is $A(\beta)$ -harmonic if and only if $\rho(A(\beta)^C) = 1$ and $\rho(A(\beta)^{\overline{C} \setminus C}) < 1$. In view of Lemma 4.2 the first condition is equivalent to $F(\mu)$ being strictly positive for every loop μ in C and that $\beta_C = \beta$. Note that $\rho(A(\beta_C)^{\overline{C} \setminus C}) = 0$ when $\overline{C} \setminus C$ is non-empty, but does not contain any components, while

$$\rho(A(\beta_C)^{\overline{C} \setminus C}) = \max \left\{ \rho(A(\beta_C)^{C'}) : C' \text{ a component in } \overline{C} \setminus C \right\}$$

otherwise. In view of i) in Lemma 4.2 and Lemma 4.3 this shows that the second condition,

$$\rho(A(\beta_C)^{\overline{C} \setminus C}) < 1,$$

holds if and only if $F(\mu) > 0$ for every loop μ in $\overline{C} \setminus C$ and $\beta_{C'} < \beta_C$ for every component in $\overline{C} \setminus C$. \square

We consider then the circular components.

Definition 4.6. A circular component C in G is a *KMS component of positive type* when

- i) $F(\nu) = 0$ for the loop ν in C ,
- ii) $F(\mu) > 0$ for all loops μ in $\overline{C} \setminus C$, if any.

Similarly, a circular component C in G is a *KMS component of negative type* when

- i) $F(\nu) = 0$ for the loop ν in C , and
- ii) $F(\mu) < 0$ for all loops μ in $\overline{C} \setminus C$, if any.

Unlike non-circular components, a circular component C can be a KMS component of both positive and negative type. This occurs when there are no loops in $\overline{C} \setminus C$.

Let C be a circular component. Assume that C is a KMS component of positive type. If there are no components in $\overline{C} \setminus C$, it follows $\rho(A(\beta)^{\overline{C} \setminus C}) = 0$ for all $\beta \in \mathbb{R}$ and we set $I_C = \mathbb{R}$ in this case. Otherwise, set $I_C =]\beta_C, \infty[$, where

$$\beta_C = \max \{ \beta_{C'} : C' \text{ a component in } \overline{C} \setminus C \}.$$

Assume then that C is a KMS component of negative type. If there are no components in $\overline{C} \setminus C$, we set $I_C = \mathbb{R}$. Otherwise, set $I_C =]-\infty, \beta_C[$, where

$$\beta_C = \min \{ \beta_{C'} : C' \text{ a component in } \overline{C} \setminus C \}.$$

In analogy with Lemma 4.5 we have the following.

- Lemma 4.7.** i) *Let $\beta > 0$. A circular component C is $A(\beta)$ -harmonic if and only if C is a KMS component of positive type and $\beta \in I_C$.*
 ii) *Let $\beta < 0$. A circular component C is $A(\beta)$ -harmonic if and only if C is a KMS component of negative type and $\beta \in I_C$.*

Proof. Basically the same as for Lemma 4.5. □

4.2. $A(\beta)$ -summable sinks.

Definition 4.8. A sink s in G is a *KMS sink of positive type* when $F(\mu) > 0$ for every loop μ in $\overline{\{s\}}$, if any, and a *KMS sink of negative type* when $F(\mu) < 0$ for every loop μ in $\overline{\{s\}}$, if any.

When there are no loops in $\overline{\{s\}}$ we set $I_s = \mathbb{R}$. When s is a KMS sink of positive type with components in $\overline{\{s\}}$, we set $I_s =]\beta_s, \infty[$ where

$$\beta_s = \max \left\{ \beta_{C'} : C' \text{ a component in } \overline{\{s\}} \right\}.$$

Similarly, when s is a KMS sink of negative type with components in $\overline{\{s\}}$, we set $I_s =]-\infty, \beta_s[$ where

$$\beta_s = \min \left\{ \beta_{C'} : C' \text{ a component in } \overline{\{s\}} \right\}.$$

- Lemma 4.9.** i) *Let $\beta > 0$. A sink s in G is $A(\beta)$ -summable if and only if s is a KMS sink of positive type and $\beta \in I_s$.*
 ii) *Let $\beta < 0$. A sink s in G is $A(\beta)$ -summable if and only if s is a KMS sink of negative type and $\beta \in I_s$.*

Proof. Left to the reader. □

4.3. The gauge invariant β -KMS states, $\beta \neq 0$. For $\beta \in \mathbb{R} \setminus \{0\}$, let $\mathcal{C}(\beta)$ be the set of non-circular KMS components C such that $\beta_C = \beta$, and $\mathcal{Z}(\beta)$ the set of circular KMS components D such that $\beta \in I_D$. Let $\mathcal{S}(\beta)$ be the set of KMS sinks s with $\beta \in I_s$. We can then summarise our findings with regard to the gauge invariant KMS states as follows.

Theorem 4.10. *Let $\beta \in \mathbb{R} \setminus \{0\}$. For every gauge invariant β -KMS state φ for α^F there are unique functions $f : \mathcal{C}(\beta) \rightarrow [0, 1]$, $g : \mathcal{Z}(\beta) \rightarrow [0, 1]$ and $h : \mathcal{S}(\beta) \rightarrow [0, 1]$ such that $\sum_C f(C) + \sum_D g(D) + \sum_s h(s) = 1$ and*

$$\varphi(S_\mu S_\nu^*) = \delta_{\mu, \nu} e^{-\beta F(\mu)} \phi_{r(\mu)}$$

for all finite paths μ, ν , where $\phi \in [0, \infty)^V$ is the vector

$$\phi_v = \sum_{C \in \mathcal{C}(\beta)} f(C) \phi_v^C + \sum_{D \in \mathcal{Z}(\beta)} g(D) \phi_v^D + \sum_{s \in \mathcal{S}(\beta)} h(s) \phi_v^s.$$

5. INCLUDING THE KMS STATES THAT ARE NOT GAUGE INVARIANT

To handle KMS states that are not gauge invariant we draw on the results of Neshveyev, [N]. For this it is necessary to introduce the groupoid picture of $C^*(G)$.

Originally graph C^* -algebras were introduced using groupoids, [KPRR], but only for row-finite graphs without sinks. For general graphs the realization as a groupoid C^* -algebra was obtained by A. Paterson in [Pa]. To describe the groupoid for a general graph, possibly infinite but countable, let $P_f(G)$ and $P(G)$ denote the set of finite and infinite paths in G , respectively. The range and source maps, r and s on edges, extend in the natural way to $P_f(G)$; the source map also to $P(G)$. A vertex $v \in V$ will be considered as a finite path of length 0 and we set $r(v) = s(v) = v$ when v is considered as an element of $P_f(G)$. Let V_∞ be the set of vertexes v that are either sinks, or infinite emitters in the sense that $s^{-1}(v)$ is infinite. The unit space Ω_G of \mathcal{G} is the union $\Omega_G = P(G) \cup Q(G)$, where

$$Q(G) = \{p \in P_f(G) : r(p) \in V_\infty\}$$

is the set of finite paths that terminate at a vertex in V_∞ . In particular, $V_\infty \subseteq Q(G)$ because vertexes are considered to be finite paths of length 0. For any $p \in P_f(G)$, let $|p|$ denote the length of p . When $|p| \geq 1$, set

$$Z(p) = \{q \in \Omega_G : |q| \geq |p|, q_i = p_i, i = 1, 2, \dots, |p|\},$$

and

$$Z(v) = \{q \in \Omega_G : s(q) = v\}$$

when $v \in V$. When $\nu \in P_f(G)$ and F is a finite subset of $P_f(G)$, set

$$)Z_F(\nu) = Z(\nu) \setminus \left(\bigcup_{\mu \in F} Z(\mu) \right). \quad (5.1)$$

The sets $Z_F(\nu)$ form a basis of compact and open subsets for a locally compact Hausdorff topology on Ω_G .² When $\mu \in P_f(G)$ and $x \in \Omega_G$, we can define the concatenation $\mu x \in \Omega_G$ in the obvious way when $r(\mu) = s(x)$. The groupoid \mathcal{G} consists of the elements in $\Omega_G \times \mathbb{Z} \times \Omega_G$ of the form

$$(\mu x, |\mu| - |\mu'|, \mu' x),$$

for some $x \in \Omega_G$ and some $\mu, \mu' \in P_f(G)$. The product in \mathcal{G} is defined by

$$(\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x, |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y),$$

when $\mu' x = \nu y$, and the involution by $(\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)$. To describe the topology on \mathcal{G} , let $Z_F(\mu)$ and $Z_{F'}(\mu')$ be two sets of the form (5.1) with $r(\mu) = r(\mu')$. The topology we shall consider has as a basis the sets of the form

$$\{(\mu x, |\mu| - |\mu'|, \mu' x) : \mu x \in Z_F(\mu), \mu' x \in Z_{F'}(\mu')\}. \quad (5.2)$$

With this topology \mathcal{G} becomes an étale second countable locally compact Hausdorff groupoid and we can consider the reduced C^* -algebra $C_r^*(\mathcal{G})$ as in [Re]. As shown by Paterson in [Pa] there is an isomorphism $C^*(G) \rightarrow C_r^*(\mathcal{G})$ which sends S_e to 1_e , where 1_e is the characteristic function of the compact and open set

$$\{(ex, 1, r(e)x) : x \in \Omega_G\} \subseteq \mathcal{G},$$

²Since we here deal with finite graphs where there are no infinite emitters, the topology has as an alternative basis the sets $Z(\nu)$, corresponding to $Z_F(\nu)$ with $F = \emptyset$.

and P_v to 1_v , where 1_v is the characteristic function of the compact and open set

$$\{(vx, 0, vx) : x \in \Omega_G\} \subseteq \mathcal{G}.$$

In the following we use the identification $C^*(G) = C_r^*(\mathcal{G})$ and identify Ω_G with the unit space of \mathcal{G} via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$. In this way we get a canonical embedding $C(\Omega_G) \subseteq C^*(G)$ and there is a conditional expectation $P : C^*(G) \rightarrow C(\Omega_G)$ defined such that

$$P(f)(x) = f(x, 0, x)$$

when $f \in C_c(\mathcal{G})$, cf. [Re]. This conditional expectation can be used to characterise the gauge invariant KMS states because it follows from Theorem 2.2 in [Th2] that a KMS state for α^F is gauge invariant if and only if it factorises through P .

To describe the automorphism group α^F in the groupoid picture we define a continuous homomorphism $c_F : \mathcal{G} \rightarrow \mathbb{R}$ by

$$c_F(ux, |u| - |u'|, u'x) = F(u) - F(u').$$

The automorphism group α^F on $C_r^*(\mathcal{G})$ is then defined such that

$$\alpha_t^F(f)(\gamma) = e^{itc_F(\gamma)} f(\gamma)$$

when $f \in C_c(\mathcal{G})$, cf. [Re].

Thanks to this picture of $C^*(G)$ and α^F , and because we consider finite graphs in this paper, we can draw on the results of Neshveyev, [N], to obtain a decomposition of the KMS states into those that are gauge invariant and those that are not. Since the groupoid \mathcal{G} has the additional properties required in Section 2 of [Th1] we can use the description obtained in Theorem 2.4 of [Th1] when $\beta \neq 0$. Of the β -KMS states considered in Theorem 2.4 in [Th1], it is only those of the form $\omega_{\mathcal{O}}^{\beta}$ which may not factor through P . Here \mathcal{O} is an orbit in Ω_G under the canonical action of the groupoid \mathcal{G} on its unit space, and \mathcal{O} must be consistent and β -summable for $\omega_{\mathcal{O}}^{\beta}$ to be defined. Furthermore, the formula for $\omega_{\mathcal{O}}^{\beta}$ shows that it is only if the points in \mathcal{O} have non-trivial isotropy group in \mathcal{G} that $\omega_{\mathcal{O}}^{\beta}$ does not factor through P .

Note that the isotropy group $\mathcal{G}_x^x \subseteq \mathcal{G}$ of an element $x \in \Omega_G$ is trivial unless x is an infinite path in G which is pre-period. Its orbit under \mathcal{G} is then the orbit of an infinite periodic path. We may therefore assume that there is a loop δ in G such that $x = \delta^\infty \in P(G)$. Then

$$\mathcal{G}_x^x = \{(x, kp, x) : k \in \mathbb{Z}\},$$

where p is the period of δ^∞ . We may assume that $p = |\delta|$ and find then that $c_F(x, kp, x) = kF(\delta)$. It follows that the \mathcal{G} -orbit $\mathcal{G}x$ is consistent in the sense used in [Th1] if and only if $F(\delta) = 0$. If the component of G containing δ contains a second loop, there will be another loop δ' in G starting and ending at the same vertex as δ . Then

$$x_n = \delta^n \delta' \delta^\infty, \quad n \in \mathbb{N},$$

are distinct elements in $\mathcal{G}x$, and when we use the notation from [Th1], we have that

$$l_x(x_n) = e^{-F(\delta')}.$$

This shows that

$$\sum_{z \in \mathcal{G}x} l_x(z)^\beta = \infty$$

for all $\beta \in \mathbb{R}$, and we conclude therefore that $\mathcal{G}x$ is not β -summable for any $\beta \in \mathbb{R}$. It follows that the only \mathcal{G} -orbits of elements with non-trivial isotropy groups which can be both consistent and β -summable in the sense of [Th1], are the \mathcal{G} -orbits of a periodic infinite path lying in a circular component consisting of a loop δ with $F(\delta) = 0$. On the other hand, for such an infinite path x the corresponding \mathcal{G} -orbit will be β -summable if and only if

$$\sum_{\mu \in E_\delta^* s(x)} e^{-\beta F(\mu)} < \infty, \quad (5.3)$$

where $E_\delta^* s(x)$ denotes the set of finite paths μ in G that terminate at $s(x) \in V$ and do not contain δ . Note that (5.3) will hold if and only if C is a circular KMS component with $\beta \in I_C$. In this case the β -KMS state ω_C^β is defined for every state φ on $C^*(\mathcal{G}_x)$, but it will only be extremal when φ is a pure state. By using the identification $C^*(\mathcal{G}_x) = C(\mathbb{T})$ this means that the extremal β -KMS states occurring in Theorem 2.4 in [Th1] that are not gauge invariant arise from a number $\lambda \in \mathbb{T}$, considered as a pure state on $C(\mathbb{T})$, and a component C of zero type with $\beta \in I_C$. We will denote this extremal β -KMS state by ω_C^λ . The formula for this state, as it was given in [Th1], becomes

$$\omega_C^\lambda(f) = \left(\sum_{\nu \in E_\delta^* s(x)} e^{-\beta F(\nu)} \right)^{-1} \sum_{k \in \mathbb{Z}} \sum_{\mu \in E_\delta^* s(x)} \lambda^k e^{-\beta F(\mu)} f(\mu x, kp, \mu x) \quad (5.4)$$

when $f \in C_c(\mathcal{G})$. A general state φ on $C(\mathbb{T})$ is given by integration against a Borel probability measure μ on \mathbb{T} and the corresponding β -KMS state ω_C^β from [Th1], which we in the present setting will denote by ω_C^μ , is then given as an integral

$$\omega_C^\mu(a) = \int_{\mathbb{T}} \omega_C^\lambda(a) d\mu(\lambda). \quad (5.5)$$

The conclusions we need here can then be summarised in the following way.

Lemma 5.1. *Let $\beta \in \mathbb{R} \setminus \{0\}$. For every β -KMS state φ for α^F there is a Borel probability measure ν on Ω_G , Borel probability measures $\mu_D, D \in \mathcal{Z}(\beta)$, on \mathbb{T} and numbers t and $t_D, D \in \mathcal{Z}(\beta)$, in $[0, 1]$ such that $t + \sum_{D \in \mathcal{Z}(\beta)} t_D = 1$ and*

$$\varphi(a) = t \int_{\Omega_G} P(a) d\nu + \sum_{D \in \mathcal{Z}(\beta)} t_D \omega_D^{\mu_D}(a). \quad (5.6)$$

The numbers t and t_D are uniquely determined by φ , as are the Borel probability measures μ_D with $t_D > 0$.

The measure ν in Lemma 5.1 have certain properties which reflect that φ is a KMS state, and they can be found in [Th1], but what matters here is only that

$$a \mapsto \int_{\Omega_G} P(a) d\nu$$

is β -KMS state which is gauge invariant. It is therefore a convex combination of the states $\varphi_C, \varphi_s, \varphi_D$ given by the formula (2.4) when the vector ψ occurring there is substituted by the $A(\beta)$ -almost harmonic vectors $\phi^C, C \in \mathcal{C}(\beta), \phi^s, s \in \mathcal{S}(\beta)$, and $\phi^D, D \in \mathcal{Z}(\beta)$, respectively. Note that the state φ_D corresponding to a component $D \in \mathcal{Z}(\beta)$ is the same as the state ω_D^m from (5.5) when m is the normalized Lebesgue

measure on \mathbb{T} . We can therefore now use Theorem 2.4 in [Th1] and combine Lemma 5.1 with Theorem 4.10 to obtain the following description of the β -KMS states when $\beta \neq 0$.

Theorem 5.2. *For $\beta \in \mathbb{R} \setminus \{0\}$,*

- *let $\mathcal{C}(\beta)$ be the set of non-circular KMS components C in G with $\beta_C = \beta$,*
- *let $\mathcal{S}(\beta)$ be the set of KMS sinks s in G with $\beta \in I_s$, and*
- *let $\mathcal{Z}(\beta)$ be the set of circular KMS components D with $\beta \in I_D$.*

For every β -KMS state φ for α^F there are numbers $\alpha_C \in [0, 1]$, $C \in \mathcal{C}(\beta)$, $\alpha_s \in [0, 1]$, $s \in \mathcal{S}(\beta)$, and $\alpha_D \in [0, 1]$, $D \in \mathcal{Z}(\beta)$, and Borel probability measures μ_D , $D \in \mathcal{Z}(\beta)$, on \mathbb{T} , such that $\sum_C \alpha_C + \sum_s \alpha_s + \sum_D \alpha_D = 1$, and

$$\varphi = \sum_{C \in \mathcal{C}(\beta)} \alpha_C \varphi_C + \sum_{s \in \mathcal{S}(\beta)} \alpha_s \varphi_s + \sum_{D \in \mathcal{Z}(\beta)} \alpha_D \omega_D^{\mu_D}.$$

The numbers $\alpha_C, \alpha_s, \alpha_D$ are uniquely determined by φ , as are the Borel probability measures μ_D for the components $D \in \mathcal{Z}(\beta)$ with $\alpha_D > 0$.

5.1. Trace states. We need a different approach when $\beta = 0$. Since the 0-KMS states are the trace states of $C^*(G)$ we must determine these.

Let $\mathcal{Z}(0)$ denote the set of circular components C in G with the property that $\overline{C} \setminus C$ does not contain any components, and similarly $\mathcal{S}(0)$ the set of sinks s in G such that $\overline{\{s\}} \setminus \{s\}$ does not contain a component. For every $C \in \mathcal{Z}(0)$ the set $V \setminus \overline{C}$ is hereditary and saturated, and there is a surjective $*$ -homomorphism $\pi_C : C^*(G) \rightarrow C^*(\overline{C})$, where \overline{C} is considered as a directed graph with vertex set $\overline{C} \subseteq V$ and the edge set $\{e \in E : s(e), r(e) \in \overline{C}\}$, cf. Theorem 4.1 in [BPRS]. Similarly, when $s \in \mathcal{S}(0)$ there is also a surjective $*$ -homomorphism $\pi_s : C^*(G) \rightarrow C^*(\overline{\{s\}})$, where $\overline{\{s\}}$ is considered as a directed graph with vertex set $\overline{\{s\}} \subseteq V$ and the edge set $\{e \in E : s(e), r(e) \in \overline{\{s\}}\}$.

When $s \in \mathcal{S}(0)$ we let n_s be the number of paths in G terminating at s . When $C \in \mathcal{Z}(0)$ we choose a vertex $v_C \in C$ and set

$$n_C = \# \{ \mu \in P_f(G) : r(\mu) = v_C, s(\mu_i) \neq v_C, \text{ for } i \leq |\mu| \},$$

where the condition that $s(\mu_i) \neq v_C$ is negligible when $|\mu| = 0$.

Theorem 5.3. *For every $s \in \mathcal{S}(0)$,*

$$C^*(\overline{\{s\}}) \simeq M_{n_s}(\mathbb{C}),$$

and for every $C \in \mathcal{Z}(0)$,

$$C^*(\overline{C}) \simeq M_{n_C}(C(\mathbb{T})), \quad C \in \mathcal{Z}(0).$$

For every trace state ω on $C^(G)$ there are unique numbers $\alpha_s \in [0, 1]$ and $\alpha_C \in [0, 1]$, and trace states ω_s on $C^*(\overline{\{s\}})$ and ω_C on $C^*(\overline{C})$, $s \in \mathcal{S}(0)$, $C \in \mathcal{Z}(0)$, such that*

$$\sum_{s \in \mathcal{S}(0)} \alpha_s + \sum_{C \in \mathcal{Z}(0)} \alpha_C = 1$$

and

$$\omega = \sum_{s \in \mathcal{S}(0)} \alpha_s \omega_s \circ \pi_s + \sum_{C \in \mathcal{Z}(0)} \alpha_C \omega_C \circ \pi_C.$$

For the proof of Theorem 5.3 set

$$N = V \setminus \left(\bigcup_{C \in \mathcal{Z}(0)} \overline{C} \cup \bigcup_{s \in \mathcal{S}(0)} \overline{\{s\}} \right).$$

Then N is hereditary and saturated, and the set $\{P_v : v \in N\}$ generates an ideal I_N in $C^*(G)$ such that $C^*(G)/I_N \simeq C^*(\tilde{G})$ where \tilde{G} is the graph with vertex set

$$\tilde{V} = \bigcup_{C \in \mathcal{Z}(0)} \overline{C} \cup \bigcup_{s \in \mathcal{S}(0)} \overline{\{s\}}$$

and edge set $\tilde{E} = \{e \in E : r(e) \notin N\}$, cf. Theorem 4.1 in [BPRS].

Lemma 5.4. *Let ω be a trace state on $C^*(G)$. Then $\omega(I_N) = 0$.*

Proof. It suffices to show that $\omega(P_v) = 0$ when $v \in N$. To this end consider a loop μ in G with vertexes v_1, \dots, v_n, v_1 . The Cuntz-Krieger relations (2.1) imply:

$$\begin{aligned} \omega(P_{v_1}) &= \omega\left(\sum_{e \in s^{-1}(v_1)} S_e S_e^*\right) = \sum_{e \in s^{-1}(v_1)} \omega(S_e^* S_e) = \sum_{e \in s^{-1}(v_1)} \omega(P_{r(e)}) \geq \omega(P_{v_2}) \\ &= \sum_{e \in s^{-1}(v_2)} \omega(P_{r(e)}) \geq \omega(P_{v_3}) = \dots \geq \omega(P_{v_n}) = \sum_{e \in s^{-1}(v_n)} \omega(P_{r(e)}) \geq \omega(P_{v_1}). \end{aligned}$$

Hence we must have equality everywhere, which implies that $\omega(P_{r(e)}) = 0$ if $e \in s^{-1}(v_i)$ for some i , but $e \notin \mu$. It follows from this that $\omega(P_w) = 0$ when

$$w \in \bigcup_{C \in \mathcal{C}} \hat{C} \setminus \bigcup_{C' \in \mathcal{Z}(0)} C'$$

where \mathcal{C} is the set of components. Hence if s is a sink in G it follows that $\omega(P_s) = 0$ unless $s \in \mathcal{S}(0)$. Consider a vertex $v \in N$. If v is sink, $\omega(P_v) = 0$ and we are done. Otherwise, if $\omega(P_v) > 0$, the Cuntz-Krieger relations (2.1) implies that there is an edge $e_1 \in s^{-1}(v)$ such that $\omega(P_{r(e_1)}) > 0$. Then $r(e_1)$ can not be a sink and we can find an edge e_2 such that $s(e_2) = r(e_1)$ and $\omega(P_{r(e_2)}) > 0$. We can continue this construction of edges e_i indefinitely so there are $i < i'$ such that $s(e_i) = r(e_{i'})$, and the path $e_i e_{i+1} \dots e_{i'}$ is contained in a component C . Since $\omega(P_{r(e_i)}) > 0$ this component must be circular and without components in $\overline{C} \setminus C$, which contradicts that $v \in N$. It follows that $\omega(P_v) = 0$. \square

For each $C \in \mathcal{Z}(0)$, fix a vertex $v_C \in C$, and set $v_s = s$ for $s \in \mathcal{S}(0)$. For all $v \in \tilde{V}$ and $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$, we define:

$$N_v^a = \{\mu \in P_f(\tilde{G}) \mid s(\mu) = v, r(\mu) = v_a, s(\mu_i) \neq v_a \text{ for } i \leq |\mu|\}$$

where the condition that $s(\mu_i) \neq v_a$ is negligible when $|\mu| = 0$. We define $N^a = \bigcup_{v \in \tilde{V}} N_v^a$ for $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$.

Lemma 5.5.

$$C^*(\tilde{G}) \simeq \left(\bigoplus_{s \in \mathcal{S}(0)} M_{\#N^s}(\mathbb{C}) \right) \oplus \left(\bigoplus_{C \in \mathcal{Z}(0)} M_{\#N^C}(C(\mathbb{T})) \right)$$

Proof. For $a \in \mathcal{Z}(0) \cup \mathcal{S}(0)$, let $e_{\alpha,\beta}, \alpha, \beta \in N^a$ be the standard matrix units in $M_{N^a}(\mathbb{C}) \simeq M_{\#N^a}(\mathbb{C})$. For $v \in \tilde{V}$, set

$$\tilde{P}_v = \sum_{a \in \mathcal{Z}(0) \cup \mathcal{S}(0)} \sum_{\alpha \in N_v^a} e_{\alpha,\alpha}.$$

Then $\tilde{P}_v, v \in \tilde{V}$, are mutually orthogonal projections. For each $f \in \tilde{E}$ such that $s(f) \notin \{v_a : a \in \mathcal{Z}(0) \cup \mathcal{S}(0)\}$, set

$$\tilde{S}_f = \sum_{a \in \mathcal{Z}(0) \cup \mathcal{S}(0)} \sum_{\alpha \in N_{r(f)}^a} e_{f\alpha,\alpha}.$$

If $s(f) \in \{v_a : a \in \mathcal{Z}(0) \cup \mathcal{S}(0)\}$, then $s(f) = v_C$ for some $C \in \mathcal{Z}(0)$, and we let μ^C denote the unique shortest path in \tilde{G} with $s(\mu^C) = r(f)$ and $r(\mu^C) = v_C$. We define an element

$$\tilde{S}_f \in C(\mathbb{T}, M_{N^C}(\mathbb{C}))$$

such that

$$\tilde{S}_f(z) = z e_{v_C, \mu^C}.$$

It is straightforward to verify that $\tilde{P}_v, v \in \tilde{V}$, and $\tilde{S}_f, f \in \tilde{E}$, is a Cuntz-Krieger family, i.e. they satisfy (2.1) relative to \tilde{G} . Since

$$\tilde{P}_v, \tilde{S}_f \in \left(\bigoplus_{s \in \mathcal{S}(0)} M_{\#N^s}(\mathbb{C}) \right) \oplus \left(\bigoplus_{C \in \mathcal{Z}(0)} M_{\#N^C}(C(\mathbb{T})) \right)$$

for all $v \in \tilde{V}$ and all $f \in \tilde{E}$, the universal property of $C^*(\tilde{G})$ gives us a canonical $*$ -homomorphism

$$C^*(\tilde{G}) \rightarrow \left(\bigoplus_{s \in \mathcal{S}(0)} M_{\#N^s}(\mathbb{C}) \right) \oplus \left(\bigoplus_{C \in \mathcal{Z}(0)} M_{\#N^C}(C(\mathbb{T})) \right).$$

To show that this is an isomorphism, note first that it is surjective because the target algebra is generated as a C^* -algebra by $\tilde{P}_v, v \in \tilde{V}$, and $\tilde{S}_f, f \in \tilde{E}$. For the injectivity we shall appeal to the gauge-invariant uniqueness theorem, Theorem 2.1 in [BPRS]. For a $a \in \mathcal{S}(0) \cup \mathcal{Z}(0)$, define for each $\omega \in \mathbb{T}$ the unitary:

$$U_\omega^a = \sum_{\alpha \in N^a} \omega^{|\alpha|} e_{\alpha,\alpha}$$

For $s \in \mathcal{S}(0)$ we define an automorphism ψ_ω^s on $M_{\#N^s}(\mathbb{C})$ by $\psi_\omega^s(A) = U_\omega^s A U_\omega^s$, and for $C \in \mathcal{Z}(0)$ we define an automorphism on $M_{\#N^C}(C(\mathbb{T}))$ by $\psi_\omega^C(f)(z) = U_\omega^C f(\omega^{\#C} z) U_\omega^C$. It is straightforward to check that:

$$\mathbb{T} \ni \omega \rightarrow \psi_\omega := \left(\bigoplus_{s \in \mathcal{S}} \psi_\omega^s \right) \oplus \left(\bigoplus_{C \in \mathcal{Z}(0)} \psi_\omega^C \right)$$

is an action, and that we for $f \in \tilde{E}$ and $v \in \tilde{V}$ have:

$$\psi_\omega(\tilde{S}_f) = \omega \tilde{S}_f \quad \psi_\omega(\tilde{P}_v) = \tilde{P}_v$$

for all $\omega \in \mathbb{T}$. It follows therefore from Theorem 2.1 in [BPRS] that the homomorphism under consideration is injective. \square

Proof of Theorem 5.3: Consider $C \in \mathcal{Z}(0)$ and let $C^*(G) \rightarrow M_{\#N^C}(C(\mathbb{T}))$ be the surjective $*$ -homomorphism obtained by composing the quotient map $C^*(G) \rightarrow C^*(\tilde{G})$ with the projection $C^*(\tilde{G}) \rightarrow M_{\#N^C}(C(\mathbb{T}))$ obtained from Lemma 5.5. The kernel of this $*$ -homomorphism is the same as the kernel of $\pi_C : C^*(G) \rightarrow C^*(\overline{C})$, namely the ideal generated by

$$\{P_v : v \notin \overline{C}\}.$$

It follows that $C^*(\overline{C}) \simeq M_{\#N^C}(C(\mathbb{T}))$. In the same way we see that $C^*(\overline{\{s\}}) \simeq M_{\#N^s}(\mathbb{C})$ when $s \in \mathcal{S}(0)$. The statements regarding a trace state ω follow from Lemma 5.4 and Lemma 5.5. \square

6. GROUND STATES

To describe the ground states we use again the groupoid picture described in Section 5 in order to adapt the approach from Section 5 in [Th4] to the present setting. The fixed point algebra of α^F is the C^* -algebra of the open sub-groupoid

$$\mathcal{F} = \{(\mu x, |\mu| - |\mu'|, \mu'x) : x \in \Omega_G, F(\mu) = F(\mu')\}$$

of \mathcal{G} . The conditional expectation

$$Q : C^*(G) \rightarrow C_r^*(\mathcal{F})$$

extending the restriction map $C_c(\mathcal{G}) \rightarrow C_c(\mathcal{F})$ can be described as a limit:

$$Q(a) = \lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \alpha_t^F(a) dt, \quad (6.1)$$

cf. the proof of Theorem 2.2 in [Th3].

When $x \in \Omega_G, z \in P_f(G)$, write $z \subseteq x$ when $1 \leq |z|$ and $x|_{[1,|z|]} = z$ or $|z| = 0$ and $z = s(x)$. An element $x \in \Omega_G$ has *minimal F -weight* when the following holds:

$$z, z' \in P_f(G), z \subseteq x, r(z') = r(z) \Rightarrow F(z') \geq F(z).$$

We denote the set of elements in Ω_G with minimal F -weight by $\text{Min}(F, G)$. Then $\text{Min}(F, G)$ is closed in Ω_G and \mathcal{F} -invariant in the sense that

$$(x, k, y) \in \mathcal{F}, x \in \text{Min}(F, G) \Rightarrow y \in \text{Min}(F, G).$$

It follows that the reduction $\mathcal{F}|_{\text{Min}(F, G)}$ of \mathcal{F} to $\text{Min}(F, G)$, defined by

$$\mathcal{F}|_{\text{Min}(F, G)} = \{(\mu x, |\mu| - |\mu'|, \mu'x) : x \in \Omega_G, F(\mu) = F(\mu'), \mu x \in \text{Min}(F, G)\},$$

is a locally compact étale groupoid. Furthermore, there is a surjective $*$ -homomorphism

$$R : C_r^*(\mathcal{F}) \rightarrow C_r^*(\mathcal{F}|_{\text{Min}(F, G)})$$

extending the restriction map $C_c(\mathcal{F}) \rightarrow C_c(\mathcal{F}|_{\text{Min}(F, G)})$. Now the proof of Theorem 5.3 in [Th4] can be repeated almost ad verbatim to yield the following.

Theorem 6.1. *The map $\omega \mapsto \omega \circ R \circ Q$ is an affine homeomorphism from the state space of $C_r^*(\mathcal{F}|_{\text{Min}(F, G)})$ onto the ground states of α^F .*

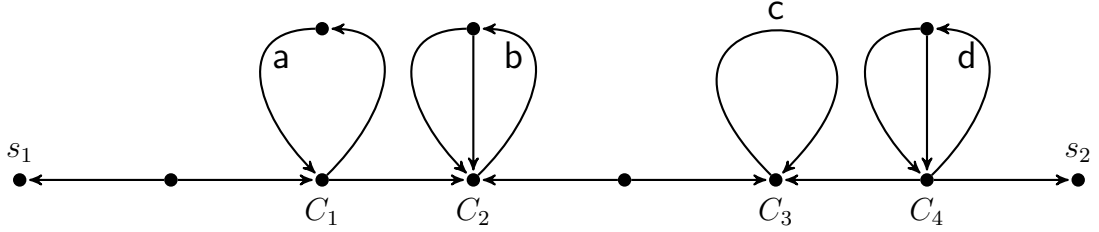
The structure of the C^* -algebra $C_r^*(\mathcal{F}|_{\text{Min}(F, G)})$ varies a lot with the choice of F . When F is constant zero, it is equal to $C^*(G)$, and when F is strictly positive it is isomorphic to \mathbb{C}^n , where n is the number of sinks in G . If G consists of three edges, e_i , and a vertex v with $r(e_i) = s(e_i) = v, i = 1, 2, 3$, and if $F(e_1) = F(e_2) = 0$ while

$F(e_3) = 1$, we find that $C^*(G)$ is the Cuntz-algebra O_3 while $C_r^*(\mathcal{F}|_{\text{Min}(F,G)})$ is a copy of O_2 .

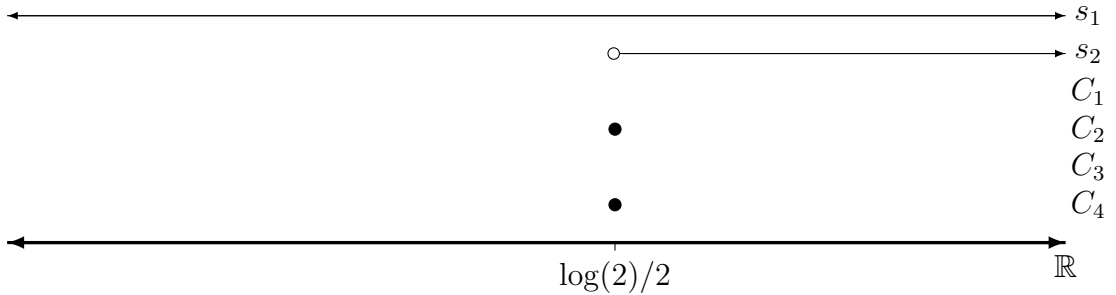
Which of the ground states are weak* limits, for $\beta \rightarrow \infty$, of β -KMS states, can be decided by combining Theorem 6.1 with Theorem 5.2. It follows, for instance, that they all are when $F = 1$, while none of them are in the last mentioned example.

7. AN EXAMPLE

Consider the following graph G . The two sinks are s_1 and s_2 and there are four components labelled C_1 through C_4 . In order to define various functions on the edge set we have labelled four edges a, b, c and d .

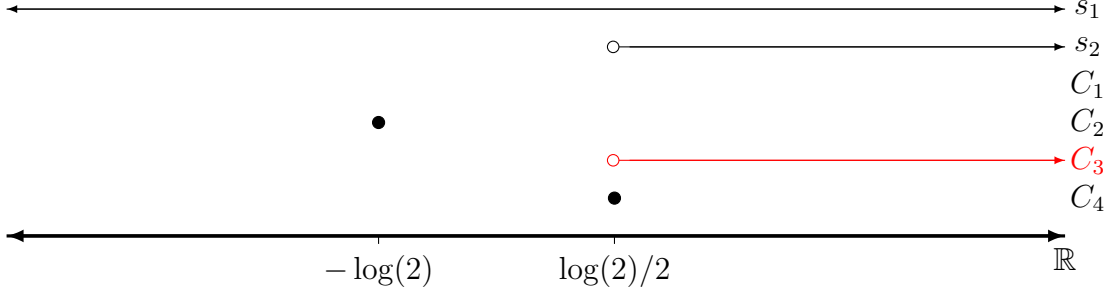


Consider first the gauge action where $F(e) = 1$ for all edges e . The two sinks are both KMS sinks in this case; with intervals $I_{s_1} = \mathbb{R}$ and $I_{s_2} =]\frac{\log 2}{2}, \infty[$. Of the components it is only C_2 and C_4 that are KMS components, both of positive type and with $\beta_{C_2} = \beta_{C_4} = \frac{\log 2}{2}$. There are three extremal β -KMS states when $\beta = \frac{\log 2}{2}$, coming from s_1, C_2 and C_4 , one when $\beta < \frac{\log 2}{2}$, coming from s_1 , and two when $\beta > \frac{\log 2}{2}$, coming from s_1 and s_2 . This 'KMS spectrum' away from 0 can be described by the following figure.



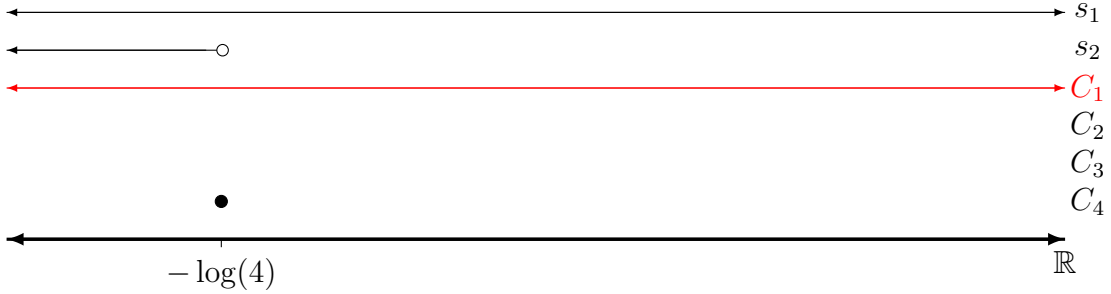
KMS spectrum ($\beta \neq 0$) for the gauge action on $C^(G)$.*

To define a different generalized gauge action, let E be the set of edges in G , and set $F_1(e) = 1$ when $e \in E \setminus \{a, b, c\}$ while $F_1(a) = F_1(b) = -2$ and $F_1(c) = 0$. If we describe the KMS-spectrum for the action α^{F_1} by a diagram as was done for the gauge action, the picture becomes the following. The red line describes the contribution from the circular KMS component C_3 and hence each point on it represents a family of extremal KMS states parametrized by a circle.



KMS spectrum ($\beta \neq 0$) for the generalized gauge action α^{F_1} on $C^(G)$.*

Finally we consider F_2 defined such that $F_2(e) = 1$ when $e \in E \setminus \{a, d\}$, $F_2(a) = -1$ and $F_2(d) = -\frac{3}{2}$. For the generalized gauge action α^{F_2} we find the following KMS spectrum.



KMS spectrum ($\beta \neq 0$) for the generalized gauge action α^{F_2} on $C^(G)$.*

The structure of the ground states vary also for the three actions. For the gauge action there are two extremal ground states coming from the sinks, while for the actions α^{F_1} and α^{F_2} there are infinitely many. Concerning α^{F_1} the sinks still contribute two, but the infinite path c^∞ has minimal F_1 -weight and contributes a family of extremal ground states naturally parametrized by a circle. The sink s_1 is the only sink which gives rise to an extremal ground state for the action α^{F_2} , but now the loop of period 2 beginning with the edge a is an element of $\text{Min}(F_2, G)$ and gives rise to a family of extremal ground states naturally parametrized by a circle.

The 0-KMS states are of course the same for all three actions. They are the trace states on the algebra, and by using Theorem 5.3 we see that they can be identified with the trace states on $M_2(\mathbb{C}) \oplus M_3(C(\mathbb{T}))$, where the sink s_1 is responsible for the first summand and the component C_1 for the second.

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